An Exploration of Regularities in Mathematics Education Pre-service Students' Responses to Variation in Number Sequences

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Abstract

The study of number patterns is an essential part of learning mathematics; however, the shift required from noticing patterns to expressing the patterns symbolically may require sophisticated algebraic techniques. The study reported on in this article focused on a group of 57 pre-service mathematics students in order to explore their interpretations of the mathematical symbolism embedded in pattern descriptions as well as their proficiency in using this symbolism to generate descriptions of the patterns. Four tasks were designed in line with Mason's theory of variation which asserts that carefully structured variation within learning activities can be used to enhance learning. The results show that although students were able to produce correct responses to the more direct questions, some students could not handle the added dimensions of variation. The study identified different strategies used by the students to reduce the elements of variation. Many students were unable to generate terms of the sequence which contained repeating cycles, and also struggled to generate a description of the general term of such sequences. It is recommended that such types of sequences may require additional scaffolding especially with respect to the use of the modulo *n* function.

Keywords: mathematics education, patterns, dimensions of variation, sequences, modulo *n*, pre-service teachers

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Introduction

Noticing generalisation and then attempting to describe these regularities lies at the heart of much of mathematics. In fact, Devlin (1997) has described mathematics as the study of patterns real or imagined, visual or mental, arising from the natural world or from within the human mind.

Experimenting with number patterns constitutes an important resource activity in developing mathematical reasoning. Asking different questions about the patterns enables the student to work at different levels of mathematical thinking. It can also provide learners with the opportunity to engage with mathematical thinking in the sense that the processes we engage with when we study patterns is a reflection of what mathematicians do when they study mathematics. The process of observing and describing patterns using mathematical notation is therefore a fundamental experience in learning mathematics.

It is important for learners as well as mathematics teachers to experience these activities with various types of patterns. However, since the study of patterns was incorporated into the South African school curriculum (DoE 2003) much of the mathematical activity around patterns has been centred on the algorithms that can be used to generate descriptions for the number patterns; thus the study of patterns has largely been reduced to learning and applying these algorithms. For example, with patterns whose terms can be described by linear functions there are particular rules for finding the general term. By looking at the first difference of a sequence T_n, say, with a first difference of d, then the formula for the general term is given by $T_n = dn + T_0$. As with sequences which can be described using a quadratic formula, teachers have developed numerous 'short cuts' allowing learners to find the values of a, b and c in the general expression $T_n = an^2+bn+c$. Samson (2008) provides a synthesis of various strategies that are used to generalise number patterns that can be described using the quadratic formula.

Zazkis and Liljedahl (2002) comment likewise that the predominant pattern-related activity for learners at schools is extending number sequences and finding an algebraic expression for the general term; that is, given the position of the element in the sequence, the goal is to find the corresponding element. Many studies focused on patterns have discussed issues related to this type of problem (Samson 2007; 2008; 2011; Lannin 2005; Dindyal 2007; Mason, Burton & Stacey 1985). In this study, two of the tasks (1 and 2) are also based on this type of problem, with one of them being based on a sequence of repeating cycles of length 3. A further two tasks (3 and 4) are of the type where the algebraic description of terms is provided and students are asked to generate some of the terms. The task of generating terms from a given description has not received much attention in the research literature perhaps because 'the ability to continue a pattern comes well before the ability to describe the general term' (Zazkis & Liljedahl 2002:388). Generating elements of a sequence when given a formula should be even simpler than continuing the pattern, because it involves substituting values into a formula and then computing the result, which could explain why there is limited literature on this area of generating terms from a given description. In this study, this basic task of generating terms of a sequence whose description is provided was raised in two different ways: first by using a sequence with repeating cycles and then by using a sequence whose terms were described recursively.

The purpose of this study which was carried out with a group of 57 pre-service mathematics students was to explore their interpretations of the mathematical symbolism embedded in the pattern descriptions as well as their proficiency in using this symbolism to generate descriptions of the patterns. The four tasks were designed in line with Mason's theory of variation (Scataglini-Belghitar & Mason 2011; Watson & Mason 2006) which asserts that carefully structured variation within learning activities can be used to enhance learning.

Literature Review

In their book, *Thinking Mathematically,* Mason, Burton and Stacey (1985) elaborated four processes which are central to mathematical thinking. These are specialising (turning to examples to learn about the question), generalising (moving from a few instances to making guesses about a wide class of cases), conjecturing (making a reasonable statement whose truth has not been established) and convincing (showing that the conjecture does hold). The authors used a variety of activities, many of which were investigations of patterns, to illustrate how these processes can be developed by asking pertinent questions. Since then many authors have used similar descriptions to capture some of the complexities of engaging with patterns. In his study of high school learners who were engaged in pattern identification tasks, Dindyal (2007) identified four sequential stages through which successful strategies evolved. The first was the direct modelling stage where learners used strategies such as counting, drawing or systematically listing the first few cases, similar to Mason *et al*'s. (1985) process of specialising. The second stage was the pattern identification stage, which is similar to the generalising process described by Mason *et al*. (1985). Dindyal's third stage was the proof testing stage where students tested their emerging generalisation with further cases, and this can be related to the conjecturing stage of Mason *et al.* (1985). The fourth stage in Dindyal's sequence was the generalisation stage where the successful students find the actual rule which is, of course, the stage where one needs to formalise what one has found to convince others. Samson (2012:8) drew upon these ideas and described pattern generalisation as '[resting] on an ability to *grasp* a commonality from a few elements of a sequence, and awareness that this commonality is applicable to *all* the terms of the sequence, and finally being able to use it to articulate a direct *expression* for the general term'.

Both Samson (2011) and Dindyal have cited Lee (1996) who identified three types of conceptual obstacles in generalisation: (1) perceptual obstacles related to seeing the actual pattern; (2) verbalisation obstacles which involve expressing the pattern clearly; and (3) obstacles at the symbolisation, which involves using mathematical notation skilfully to express the pattern that is observed. In trying to describe some of the symbolisation challenges experienced by students, Arcavi (2005) introduced the notion of symbol sense as an essential and multifaceted way of working with symbols in algebra. He defined symbol sense as the 'ability to manipulate and also to 'read though' symbolic expressions as two complimentary aspects' (Arcavi 2005: 43). It also includes the ability to select one possible symbolic representation of a situation and if necessary to discard it in favour of a more suitable one. Arcavi concedes that using symbols as a productive tool to investigate relationships is not easily accomplished and requires focused interventions by the mathematics educator. In this study the focus was on the problems experienced by pre-service mathematics education students at the symbolisation level, both in expressing the pattern by manipulating symbolic expressions and being able to read through and interpret the algebraic symbolism used to describe patterns.

Dindayal (2007) noted that one of the issues related to symbolisation is that students 'often focus on inappropriate aspects of a number pattern' which may side-track them from arriving at an explicit and appropriate generalisation. Another factor that may affect the success of the symbolisation process is the strategy that students use to come up with the generalisation. Hershkowitz *et al*. (2002) observed that generalisations could be expressed in terms of the step-by-step recursive method or in terms of the independent variable. Samson (2011, 2012) focused on embodied processes that learners experience when they are engaged in pattern generalisation tasks embedded in specific situations modelled by linear sequences. He found that the articulation of an algebraic generalisation is complicated by tensions between local and global visualisation. A local visualisation is similar to the recursive step-by-step method described by Hershkowitz *et al*. (2002). Generating a formula in this case involves looking at how the pattern in the new stage has changed from the existing one, by adding or subtracting a structural unit into the existing term. A global visualisation is one that tracks the behaviour of the independent variable in the different cases of the pattern setting and expresses the generalisation in terms of this behaviour. However, local generalisations are not always easier than global generalisations. An example of this will be illustrated in this article, when the formula for a pattern is expressed recursively.

Zazkis and Liljedahl (2002) explored the attempts of a group of pre-service elementary teachers to generalise a repeating number pattern. The authors found that students' ability to express generality verbally was not accompanied by and did not depend on their use of algebraic notation. There was a gap between students' ability to express generality verbally and their proficiency in using algebraic notation. The authors concede that the difficulty was related to the task itself consisting of a pattern with repeating elements which did not 'lead to a 'smooth' algebraic notation, presented in one 'neat' formula that connects the element *n* to its location' (Zazkis & Liljedahl, 2002:399). Repeating patterns have a recognisable repeating cycle of elements, and this aspect was a focus of the study by Threlfall (1999). Threlfall advocated the varying of some attributes of elements while keeping other elements constant in order to add complexity to a repeating pattern. Although the numbers themselves did not repeat in the pattern investigated by Zazkis and Liljedahl (2002), applying the same transformation to each element produced an explicitly recognisable repeating cycle. It was the position of the numbers that formed the repeating cycle of length 8, and the authors used the function modulo 8 as a tool to describe how the position of a number could be identified based on the output of the function (that is, the remainder of the division of the number by 8). It is believed that this study will extend the research in the area of symbolisation of patterns with repeating cycles.

Theoretical Framework

Being immersed in pattern-spotting and pattern-extension activities undoubtedly contributes to learning mathematics. However, Watson and Mason (2006) caution that 'learning does not take place solely though learners observing some patterns in their work' even if they have developed generalisations of the patterns. The authors comment that activities such as pattern-spotting, generalising and reproducing patterns are just the means by which people make sense of experiences and these activities need to be carefully planned and sequenced in order to facilitate meaningful learning. Watson and Mason (2006) contend that by being exposed to structured or structural experiences aimed at exposing underlying mathematical form, learners' ways of working can be shifted to higher levels. The authors identify mathematical variation as a scaffolding tool that can be used in mathematical activities to shift learners' thinking towards a more conceptual orientation. Watson and Mason (2006) focus on 'dimensions of possible variation' which refer to 'features, aspects and parameters that can be changed in an object whilst remaining an example of a concept.'

Watson and Mason have argued that by paying attention to variation in the design of tasks, a teacher can provide more structured opportunities for learning. As learners move from working with familiar and similar examples to not-quite-so-similar tasks, they are shifted to working on higher levels. Scataglini-Belghitar and Mason (2011) cite Marton and Booth (1997) who defined learning as 'extensions of dimensions of variation of which a learner is aware'. Watson and Mason (2006:97) support Martin and Booth's definition as a means for 'describing learning, for relating learning to mathematical structures as afforded to, and perceived by, learners'. Hence, the responses to structured variation provide a means of observing learning. For teachers the construction of tasks that use variation and change optimally constitutes an ongoing 'design project' which requires constant revision and changes based on the observed learning (Watson & Mason 2006:100).

Herscovics (1989) defined the term 'epistemological obstacles' as obstacles that are encountered in the development of knowledge in a discipline. Learning is enhanced as a person finds ways of overcoming the difficulty posed at that point. Didactically speaking, teachers need to identify these epistemological obstacles so that they can better help the students to move forward. Sometimes it may be the case that the approach taken by the teacher may cause further conceptual difficulties for the students. Olivier (2013) refers to 'didactical obstacles' as difficulties experienced by students which may result from the teaching approach employed by the teacher.

In the study on which this article is based I focused on the epistemological obstacles caused by the contrived introduction of mathematical variation in the context of sequences in an effort to observe the regularities evident in the ways in which the students respond to these variations. As the instructor of the module in which this study was carried out, I hoped that the identification of these regularities would inform the refinement of the study materials so that the didactical obstacles can be reduced in future offerings of the module. Hence the study will contribute in general to pre-service mathematics teacher education pedagogy.

Methodology

The study utilised an interpretive approach because the main goal of this study was to understand the students' interpretations of reality (Cohen, Manion & Morrison 2011) with respect to algebraic problems set around sequences. The participants in the study were 57 students out of a class of 59 who were enrolled on a Real Analysis course for pre-service teachers where students study topics in set theory, topology of the real line, number theory, proof, and sequences and series. The unit on sequences and series was designed to extend the students' experiences of patterns and sequences beyond arithmetic, quadratic and geometric which they encounter at school. The unit consisted of many exploratory activities which involved continuing sequences, generating terms of sequences whose *n*th term was provided and providing formulae to describe the general terms of various types of sequences. A selection of the test items in the module was specially designed both for assessment and research purposes and the four tasks were included in a larger module assessment. The analysis of the responses can be regarded as content analysis which 'simply defines the process of summarising and reporting written data and their messages' (Cohen *et al.* 2011:563). In this case the students' responses are the source of the communication intended to convey their engagement with the sequences. The analysis of students' responses to assessments or specially designed tasks for research purposes serves as a valuable resource for analysing academic activities. These data can be used to provide information about students' varying engagement with particular concepts; the competence of students in the area being assessed; the mathematical demands of the task, thus improving the conceptual understanding of the researchers doing the analysis; and possible sequencing of the teaching of particular concepts. Hence, in such studies, the work of teaching strengthens and is in turn strengthened by the work of research.

The data analysis process involved studying the responses of the 57 students with a view to understanding firstly the 'what', and then the 'why' and the 'how' underlying the data (Henning 2004). Dey (1993:30) describes data analysis as 'a process of resolving data into its constituent components to reveal its characteristic elements and structure'. In a similar manner the students' responses were broken down into constituent parts reflecting their reactions to those elements of variation identified in the problem setting. This was done in order to classify and make connections across the data elements (Henning 2004:128). The responses were coded, which means representing 'the operations by which data are broken down, conceptualised, and put together in new ways' (Strauss & Corbin 1998:120). Hence, each written response was analysed in terms of how the student reacted to the perceived dimensions of variation present in the tasks.

The research question guiding the study was: How do students respond to the dimensions of variation in tasks based on sequences?

One task required students to find a formula for the *n*th term of a sequence. The first task (Task 1) required students to generate terms given the formula of the general term. The demand of this basic task of generating terms of a sequence was then raised using two approaches. One way of raising the demand (Task 3) involved a sequence with repeating cycles, hence the position of the term was made dependent on output of the modulo 3 function. The challenge in this case seems to be that both the term and the position were now being varied. A second way in which the task of generating terms of a sequence was made more complex was by providing the description in recursive terms, and Task 4 took on this dimension. The third task required students to find a formula for the *n*th term of a sequence with repeating cycles of length 3.

The responses to four tasks are analysed in this article. A detailed discussion follows in the next section.

Tasks

Before presenting the tasks, I first discuss the definition of a sequence that was used in the real analysis module.

> A *sequence* of real numbers (or a sequence in ℝ) is a function on the set ℕ, of natural numbers whose range is contained in the set ℝ of real numbers. That is, (a_n) can be seen as a function f: ℕ → ℝ.

The general way of writing a sequence is a_1, a_2, a_3, a_4, …… a_n, ……, so that for each element there is an element of the sequence, a_n. This means that a sequence must be an infinite (not finite) list of terms, though repetition is allowed. Such a sequence is denoted by (a_n) or (x_n) or X or (x_n: $n \in$ ℕ). Note that x_n or a_n is the single number denoting a term of the sequence and is also denoted by T_n. The independent variable, *n*, marks the position of the term. For example, the fourth term means T_4 while the value of the fourth term refers to the actual value of T_4 or f(4). Sometimes students experience difficulties in distinguishing between the position *n* with the value $f(n)$ or T_n. For example, consider the sequence $T_n = f(n) = 9n + 4$. When asked: For which value of *n* is T_n equal to 40, say, some take *n* to be 40, and they find T_{40}, instead finding *n*, when $9n + 4 = 40$.

It is important to note that 'no finite sequence of numbers uniquely generates the next term' (Zazkis & Liljedahl 2002:384) and that a finite array of numbers may be extended in a variety of ways. Hence, the questions specified a possible formula for the general term and not 'the' general formula.

The details of the four tasks are now presented. In each case the salient elements of the task are discussed. The detailed interrogation of each task that is presented is necessary for the discussion of the results that follow.

Task 1

Write down the first four terms of the following sequences, and the tenth term $\frac{(-1)^n.n}{n+1}$	Solution: $x_1=-\frac{1}{2}$; $x_2=\frac{2}{3}$; $x_3=-\frac{3}{4}$; $x_4=\frac{4}{5}$; $x_{10}=\frac{10}{11}$

The mathematical skill assessed in Task 1 was the ability to calculate the terms *f*(*n*) of the sequence, given the description *f*, and the value *n*.

Task 2

Find a formula for the *nth* term of these sequences: 1, 3, 4, 1, 3, 4, 1, 3, 4, 1,3,4, ………	**Solution** $\boldsymbol{x_n} = \begin{pmatrix} 1 \ \ if\ n = 3m+1, \\ 3 \ \ if\ \ n = 3m+2 \\ 4 \ \ if\ n = 3m+3 \\ where\ m \in \mathbb{N}_0. \end{pmatrix}$

In this task, the demand for using a formula to describe the *n*th terms has been raised by making the sequence one that repeats in strings or cycles of three numbers. Hence there are three possible values taken on by terms in the sequence depending on the position of the term. The **modulo** (sometimes called **modulus**) operation which finds the remainder of division of one number by another, is used to address the issue of the repeating terms.

Given two positive numbers, *n* (the dividend) and *a* (the divisor), *n* **modulo** *a* (abbreviated as *n* mod *a*) is the remainder of the Euclidean division of *n* by *a*. For instance, the expression ‘5 mod 2’ would evaluate to 1 because 5 divided by 2 leaves a quotient of 2 and a remainder of 1, while ‘9 mod 3’ would evaluate to 0 because the division of 9 by 3 has a quotient of 3 and leaves a remainder of 0; there is nothing to subtract from 9 after multiplying 3 times 3. The operation of modulo 3 is one which can be used in this problem. The operation results in a partitioning of the set N into three subsets, corresponding to the elements which are evaluated to 0, 1 and 2 respectively by the operation *n* mod3. Note that

$$n \bmod 3 = 1 \;\; if \; n = 3m+1,$$
$$n \bmod 3 = 2 \; if \;\; n = 3m+2$$
$$n \bmod 3 = 0 \quad if \; n = 3m+3$$
$$where \; m \in \; \mathbb{N}_0.$$

For discussion purposes I will refer to the expressions $3m+1$, $3m+2$ and $3m+3$ as g_1, g_2 and g_3 respectively. Furthermore, I will refer to the subsets of ℕ as G_1, G_2 and G_3, where $\mathbb{N} = G_1 \cup G_2 \cup G_3$ and

$G_1 = \{n \in \mathbb{N} \mid n \bmod 3 = 1\} = \{1; 4; 7; 10; \ldots\}$.
$G_2 = \{ n \in \mathbb{N} \mid n \bmod 3 = 2\} = \{ 2; 5; 8; \ldots\}$
and $G_3 = \{ n \in \mathbb{N} \mid n \bmod 3 = 0\} = \{3; 6; 9; \ldots\}$

Task 3

Find the first four terms and the tenth term for the following sequence. $x_n = \begin{pmatrix} 2n, \; when \; n = 3m+1 \\ 2n-2, when \; n = 3m+2 \\ 2n-2, \; when \; n = 3m+3 \end{pmatrix}$	Solution. $x_1=2$; $x_2=2$; $x_3=4$; $x_4=8$; $x_{10}=20$

Here the function defining the sequence is made up of different rules for elements from each of the subsets G_1, G_2 and G_3. For the discussion, I will refer to the three functions that are applied on the subsets G_1, G_2 and G_3, as f_1, f_2 and f_3 where

$f_1: G_1 \rightarrow \mathbb{N}$ and $f_1(n) = 2n$ $\quad f_2: G_2 \rightarrow \mathbb{N}$ and $f_2(n) = 2n-2$; $\quad f_3: G_3 \rightarrow \mathbb{N}$ and $f_3(n) = 2n-2$

Hence the sequence can be represented as follows.

$$x_n = \begin{pmatrix} f_1(n) = 2n \text{ when } n \in G_1 \\ f_2(n) = 2n\text{-}2, \text{when } n \in G_2 \\ f_3(n) = 2n\text{-}2, \text{ when } n \in G_3 \end{pmatrix}$$

Task 4

Consider the sequence defined by x_1 = 8, and x_{n+1}= x_n + (4n-2). 1. Write down the first four terms of this sequence. 2. Find a formula to describe the nth term of this sequence.	Solution. x_1=8; x_2=10; x_3=16; x_4=26

The description of the general term given in Task 4 can be broken up into three parts:

x_{n+1}= x_n + (4n-2)

The subject of the formula — first term on the RHS — second term on the RHS

Thus the formula or function is $f(n+1) = x_n+(4n-2)$. When $n = 2$ say, the term on the LHS takes on x_3, while the first term on the RHS is x_2 and the second term of the RHS is (4(2)-2). Although the value of n is being consistently substituted, the resulting expression consists of both $x_3$and x_2, which is a dimension of variation that seems to have created some sort of unease.

Results

The students' responses to the four tasks were first categorised as either correct or incomplete or incorrect. The incomplete and incorrect responses were then analysed for emerging themes through a general inductive analysis. The initial results for the four tasks are summarised in Table 1.

Table 1: Overall results for the four tasks

Task	Number correct	Number incorrect or incomplete
1	46	11
2	31	26
3	14	43
4	33	14

As suggested by the results in Table 1, students found Task 1 least difficult while Task 3 was most challenging. The details of the results for each task follow next.

Results for Task 1

For this task, most students (46) were able to substitute the five values of n and correctly calculate the first four and tenth terms. There were 10 students who did not specify the tenth term, but they correctly presented the first four terms. One student made a slip with the negative sign on one term. Thus it can be said that all the students were able to substitute various values into the given formula to generate terms.

Results for Task 2

There were 26 students who did not produce a correct response for Task 2. The key tool for Task 2 was the use of the mod 3 function. That is, they needed to identify that the terms of the sequence were appearing in cycles of three and therefore the mod 3 function could be used to help them describe the values of terms that appeared in certain positions. These students' responses indicate different degrees of struggle with using the mod 3 function as a tool.

Some students tried to generate a general formula that could describe all the terms, such as student 29, whose response appears in Figure 3

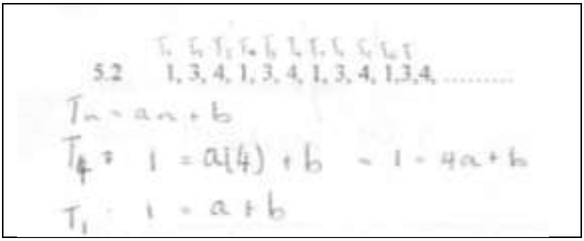
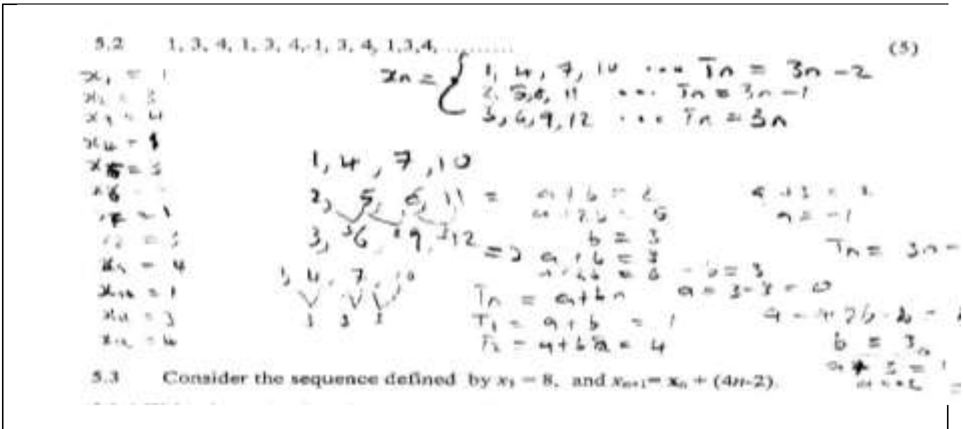


Figure 3: Response of student 29 to Task 2

Figure 3 shows that S29 tried to find one formula to represent all the terms. Perhaps this was an attempt to reduce the variation caused by the repetition of the terms cycles of three. However he did not succeed with this approach.

Other students did not ignore the fact that the terms were appearing in cycles of the three terms and recognised that the *n* mod 3 function could be used to address this issue. For example, 15 students listed the sets G_1, G_2 and G_3. The response from student 24 appears in Figure 4. The student provided a description of the elements which belonged to the three sets, as $T_n = 3n-2$, $T_n=3n-1$, $T_n=3n$.



Student 24

Figure 4: Response of S24 to Task 2

The responses in Figure 4 show that the students generated expressions to describe the elements of the sets G_1, G_2 and G_3. However, giving such a description is only a first step to the solution. A further step was to specify the values taken on by terms T_n where, n belongs to each of the Gi which these students did not get to, but presented the descriptions as formulae for the terms, T_n. The neglect of the second step indicates that the students did not distinguish between the position n and the value T_n of the terms.

Results for Task 3

There were only 14 students who produced the correct answers. I will now discuss some of the common trends behind those responses which were incorrect.

There were eight learners who listed the elements of G_1, G_2 and G_3, as shown in the response by student 7 in Figure 5.

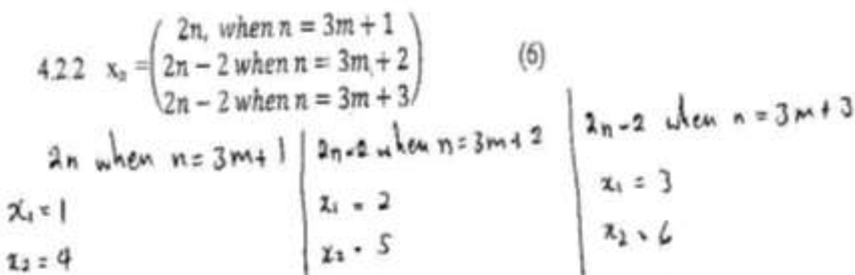


Figure 5: Student 7 response to Task 3

This student wrote the elements of the sets G_1, G_2 and G_3. The student transformed each of the expressions 3m+1, 3m+2 and 3m+3 into functions g_1, g_2 and g_3 respectively where $g_1(m) = 3m+1$, $g_2(m) = 3m+2$ and $g_3(m) – 3m+3$. That is, he systematically worked out $g_1(0) = 3(0)+1 = 1$, by substituting the values m=0, 1, 2 into the expression $3m+1$ and generated the values 1, 4, 7, ….., which are the elements of the set G_1.

He then substituted the values m=0, 1, 2 into the expression $3m+2$ (or g_2), and generated the values 2, 5, 8, etc., which are the elements of set G_2. Finally he substituted the values m=0, 1, 2 into the expression $3m+3$ (or g_3), and generated the values 3, 6, 9, etc., which are the elements of set G_3. The student's response indicates that he took the expressions g_1, g_2 and g_3 as functions for calculating the value of T_n. However these are descriptive conditions that are satisfied by certain natural numbers, and depending on which condition the number satisfies, the number becomes an input of the functions f_1, f_2 or f_3, to produce the required term value.

Some students tried to use the sets of expressions and functions in various ways. One of these students was student 3, whose response appears in Figure 6.

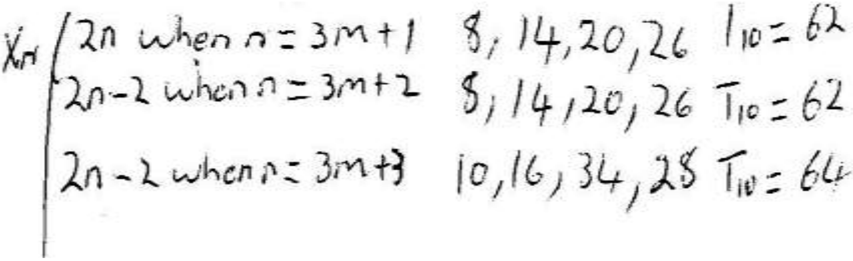
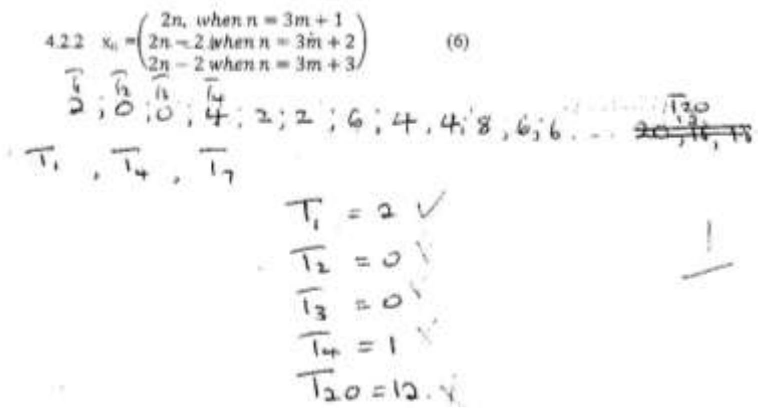


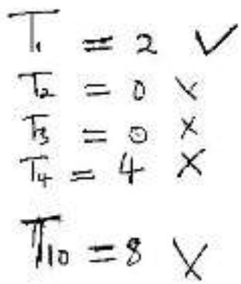
Figure 6: Response by student 3 to Task 3

It can be seen in Figure 6 that the student considered the three possibilities of f_1, f_2 and f_3 separately. She also took the expressions $3m+1$; $3m+2$ and $3m+3$ as functions g_1, g_2 and g_3. This enabled her to calculate $f_1 \circ g_1$ (n) for n=1, 2, 3, 4 and n =10 and she generated the list of values in the top row. Thereafter she found $f_2 \circ g_2$ (n) for n = 1, 2, 3, 4 and n =10 and generated the values in the middle row. Finally she found $f_3 \circ g_3$ (n) for n =1, 2, 3, 4 and n =10 and wrote the values in the bottom row. The student generated three sets of terms and she did not find a way to coordinate these sets into one sequence.

There were three students who wrote the first few terms as 2, 0, 0, 4, 2, 2, as shown in Figure 7.



Student 43



student 8

Figure 7: Responses of students 8 and 43 to Task 3

From Figure 7, it seems as if in trying to deal with the cyclic nature of the sequence these students tried to hold or pin one quantity down while allowing the other to vary. They kept n constant at n =1and found $f_1(1)$, then $f_2(1)$ and $f_3(1)$. Then they moved to n = 2 and found $f_1(2)$, then $f_2(2)$ and $f_3(2)$ and thereafter moved to n=3 found $f_1(3)$, then $f_2(3)$ and f_3. Moving on to n =4, they then calculated the tenth term using n=4, that is $T_{10} = f_1(4) = 8$. They tried to keep *n* constant while going through the three functions f_1, f_2 and f_3 in turn.

Many of the students (5) produced a repeating sequence, 1, 2, 4, 1,2 4. One of these students was student 11, whose response is shown in Figure 8.

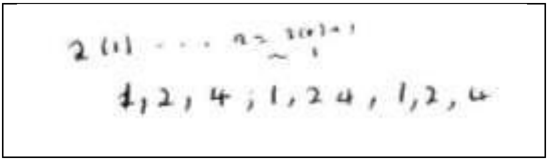


Figure 8: Response of student 11 to Task 3

This approach of repeating terms may have been influenced by the form of the sequence in Task 1.

It is important to note that all except four of the 26 students whose responses to Task 2 were incorrect did not produce a correct response to Task 3. It is possible that their response was incomplete for Task 2, but they nonetheless had an understanding of the *n* mod3 function which was also a tool used in Task 3.

The other 22 students whose responses to Task 2 were incorrect also gave an incorrect response to Task 3. That is, 85% of those who did not produce a correct response to Task 2 were unable to produce a correct response to Task 3. There were 11 students who had produced a correct description of the terms of the sequence in Task 2 but could not do the same for Task 3. It may therefore be surmised that for this sample of students Task 3 presented a greater challenge. An examination of Task 3 confirms that the demand was increased by the introduction of a further dimension of variation. Whereas for Task 2, the value of T_n was fixed at 1, 3 and 4 when *n* was an element of each of the three sets G_1, G_2 and G_3 respectively; for Task 3, this was now varied further. Instead of fixed values, T_n was now described using a different rule for elements from each of the three sets.

Results for Task 4 (Question5.3.1)

In Task 4, the additional dimension of variation is not with the addition of new variables but with the variation of the notation of the terms x_n and x_{n+1}. Whereas in the former the substitution of n results in a term x_n, for the latter the substitution of the same value of n now results in the subsequent term x_{n+1}.

There were 33 students who found their way around the three substitutions of n in the formula and who correctly computed the first four terms.

There were 11 students who produced the response 10, 16, 26, 40. An example of this is the response by S5 presented in Figure 9.

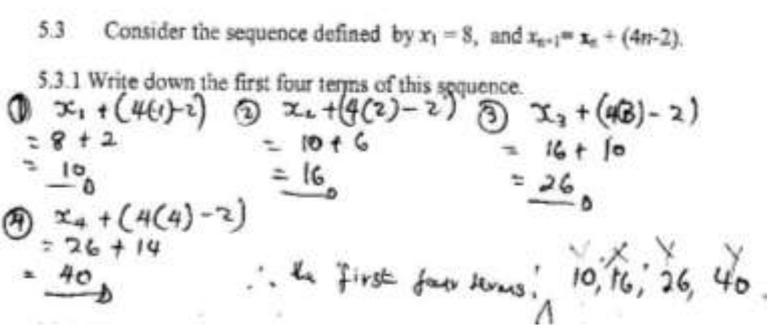
	Actual response: 10, 16, 26 40 Effective formula $x_n = x_n+4n-2$

Figure 9: Response by S5 to Task 4

These students substituted n=1 in both parts of the formulae on the RHS but implicitly took n=0 in the subject of the formula for the first term; similarly for the second term they substituted n=2 into the expressions on the RHS and implicitly took n=1 in the subject of the formula on the LHS. The response of student 5 shows that she took the first term as being different from x_1 which was given already as 8. She used the x_1 as something to be substituted into the formula to find the first term of the sequence, and did not take x_1 as the first term. Similarly, she saw the second term as being a different entity from x_2. In general, her nth term= x_n+4n-2. These approaches have actually changed the given formula to $x_n = x_n+4n-2$, which is mathematically incoherent and as an equation it has no solution. The misconception here is that n can take on different values in one formula, and also that x_1 is different from the first term.

Another common misconception is revealed in the responses of nine students, who wrote the first four terms as 8, 14, 24, 38. The response of S24 appears in Figure 10.

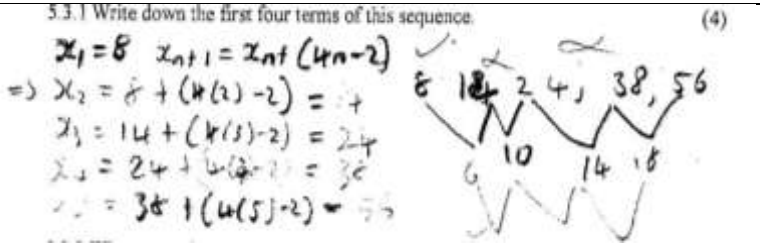
	Response 8, 14, 24, 34 Effective formula used $x_{n+1} = x_n +4(n+1)-2$

Figure 10: Response of S24 to Task 4

From Figure 10 it can be seen that for the calculation of x_2, the student took n=1, in the subject of the formula and in the first part of the formula but took n=2 in the second part of the formula. Similarly, for the calculation of the third term, n was taken as n=2 in the subject and the first part of the formula, while it was taken as n=3 in the second part, hence the effective formula was $x_{n+1} = x_n+4(n+1)-2$.

Some students used an incomplete formula by taking only the second part of the formula such as 6, 10, 14, 18 or 10, 14, 18, 22, which can be represented as $x_n= 4(n+2)-2$

One student wrote 8, 10, 12, 14, which was effectively taking $x_{n+1} = x_n+2$. Another variation in the formula was 8, 14, 18, 22, 26, which involved adding $4n$-2 to 8 and the effective formula was $x_{n+1}= 8+4n-2$.

Therefore, for Task 4, the ways in which the students used the given formulae resulted in rules that were different from the original formula. Their attempts at trying to reduce the variation resulted in substantial changes to the sequences.

Discussion

The responses to the questions show that there were often common strategies in the ways in which the students interpreted the pattern descriptions and the ways in which they generated patterns whose descriptions were provided. These regularities provide an indication of the students' interpretations and use of the symbolisation. The students' struggles may be related to similar struggles experienced with studying rates of change, where the change in one quantity is observed with respect to the change in a second quantity. Students' difficulties with examples such as acceleration as the rate of change of velocity as the rate of change of distance are well documented (Tasar 2010; Thompson & Thompson 1994; 1996). Another example of change involving two quantities is that of inflation and price of goods (Tasar 2010; Bansilal 2011). In the cases considered in this article the changes are somewhat different from those observed and described in rates of change, but relate to the discomfort of dealing with variation on different levels. Whereas with rates of change, there is one dependent variable which is changing with respect to a second independent variable. In this study, the changes were somewhat different. With Tasks 2 and 3 change was occurring on three fronts. There was variation in position, which was dependant on the value of n or term number; there was variation in conditions, where n could satisfy either of the three conditions, g_1, g_2 or $g_{3;}$ and then there was variation in the formula for computing the term value which could be f_1, f_2, or f_3. With respect to Task 4, the variation was in the term taking on the value of x_n and x_{n+1} in the same expression.

One of the demands that the changes or the variations induced was that the students' conceptions were challenged. Without the variations, substitution into a formula in mathematics may be a straightforward exercise. Usually when the formula is given, one just needs to substitute the given value and then carry out the computation using operations on numbers; in fact, almost all the students in the class were well able to deal with this demand, as revealed in their responses to Task 1. Mason (1989:4), writing in the context of learners who are being introduced to algebra, notes that learners can often express a general rule but find it difficult to see the expression as an object which can be manipulated or transformed. Mason's theory of shifts implies that 'they need continued exposure to such acts of expressing so that they begin to find it relatively easy' (Mason 1989:4) so that it almost becomes routinised, allowing them to shift their perception from seeing it as a formula to seeing it as an object. In terms of Task 4, many students were able to use x_n in the substitution to calculate the $n+1^{st}$ term, using x_n as an input into the formula. This was in following a rule as in 'to get the next term substitute the previous term and add 4*n* and subtract 2'. However, the formula required a variation in the role of x_n as moving from an input for the $n+1^{st}$ term to being an object in its own right as the result of the formula for the *n*th term. The responses show that many could not make the shift in the perspective of x_n as an object that is the *n*th term. The variation in the role of x_n thus constituted an epistemological obstacle, which if passed results in extended learning. As a person finds ways of overcoming the difficulty posed at the point of the epistemological obstacle, learning is enhanced (Herscovics 1989). Students who work with such expressions, and develop a dual perspective of x_n as an input for the $n+1^{st}$ term while also being an object, have become aware of the extension in the dimensions of variation of x_n (Scataglini-Belghitar & Mason 2011).

Mason's theory of shifting one's perception from seeing a formula as an object also adds insight into the case of the *n* mod3 function that appeared in Tasks 2 and 3. In Task 2, many students struggled to move from the expressions which specified the positions of terms to specifying the value of the terms. Some (15) were able to find a formula to describe the elements which belonged to the three sets G1, G2 and G3, or the positions of terms. They generated the three lists of *n*-values which satisfy the three expressions related to the outputs of *n* mod3 function respectively. The generation of these lists may have crystallised the operation of *n* mod3. However, they needed a further shift that would have allowed them to see the partitioning (G1, G2 and G3) as the result of the operation of *n* mod3. They struggled to shift their attention to seeing these *n*-values as objects upon which the different functions f_1, f_2 and f_3, could operate; hence the added variation of the repeated cycles of length three also constituted an epistemological obstacle. There were students (22) who did not produce correct answers for Tasks 2 and 3. This demonstrates that not making the shift from formula to object in Task 2 also hampered them in working with the repeated cycles in

Task 3. In order to perform further operations on the elements of the sets G1, G2 and G3, it was necessary to have an object-conception of the *n* mod3 function. However, the students were stuck in a process conception (Mason 1989) – they had not moved further than seeing the *n* mod 3 function as a process, so they could not perform further operations on the elements of those sets or lists.

In attempting to deal with the dimensions of variation embedded in Task 3, students seemed to be trying to find ways to keep certain quantities constant while varying others, as seen in the various responses that were presented. The introduction of one set of expressions (g_1, g_2, g_3) for checking the position and another set of functions (f_1, f_2, f_3) for evaluating the terms (T_n) introduced multiple dimensions of variation that complicated the problem. Some students responded by focusing only on the expressions g_1, g_2 and $g_{3,}$ and did not consider the functions f_1, f_2 and f_3. Others considered expressions g_1, g_2 and g_3 as functions which could operate on the *n*-values. Some considered the composition of the g_i's and f_i's to make up three different functions which led to three different sequences. Some kept *n* constant while they cycled through the various f_i, and some came up with a sequence which had repeating cycles of length 3. It is worth noting that this type of pattern with repeating cycles also proved to be challenging in other studies. Zazkis and Liljedahl (2002:399) noted that the pattern with repeating elements presented difficulties because it did not 'lead to a 'smooth' algebraic notation, presented in one 'neat' formula that connects the element *n* to its location.' In their study with 36 pre-service elementary school teachers, there were only three teachers who used a strategy related to the *n* mod 8 function supporting my finding that patterns with repeating cycles were experienced as very difficult.

Conclusion

The study reported in this article was an exploration of pre-service students' responses to induced dimensions of variation in representing sequences. These dimensions of variation presented epistemological obstacles to them which some tried to overcome by looking for ways to minimise the variation by trying to keep some aspects constant. However, the definition used by Watson and Mason (2006), that sees learning as an extension of '[dimensions of variation that a student is aware of, implies that as students overcome these epistemological obstacles their learning will be extended. It is suggested that these experiences should be brought to the fore for these students who are going to be teachers, so that they can reflect on what made them struggle and how this effort contributed to their learning. As teachers the lessons they learn from their own learning experiences may help them plan learning experiences for their own learners in future.

In terms of my own introspection, the students' struggles with the concept of the mod 3 function suggest that they may have needed additional help. The experience of encountering it for the first time in its role in generating and describing repeated cycles of terms in sequences indicates that the introduction via this route constituted a didactical obstacle (Olivier 2013). Perhaps these repeating sequences needed to be scaffolded first by introducing the mod *a* function first so that students could gain familiarity with the operation of modulo *a*. They may then see the value of the mod *a* function in representing cycles of length *a* in a sequence.

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