Making Sense of Mathematical Discourse: Implications for Success in the Learning of Differentiation in a University Classroom

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Abstract

This article reports part of a case study which investigated errors that occurred in a Mathematics classroom owing to poor conceptualisation of mathematical concepts and a lack of the required mathematical register among a group of thirty first year university students. The study focused on the critical issue of making sense of mathematical concepts and examined this issue by exploring students' interpretations and misinterpretations in their learning of mathematical differentiation. The students were registered for Mathematics 1 in Chemical Engineering. Data was derived from the students' written work, along with video-recorded material of their 'live' learning interactions. This data was then analysed within the framework of Sfard's (2007) Commognitive approach as a means of attempting to address errors that are displayed by students when learning differentiation. The students demonstrated different types of difficulties, namely, conceptual, interpretation, procedural, linear extrapolation and arbitrary errors. The application of the Commognitive approach to the identification and addressing of students' errors to advance their understanding of differentiation has proven to be positive in enhancing student learning and success.

Keywords: Commognitive framework; mathematical discourse; students' errors; students' understanding; success in mathematics

Introduction

Students' understanding of mathematics causes uneasiness among teachers, lecturers and other interested parties (Siyepu 2013a). It is somewhat difficult to predict the causes of difficulties that make several, primarily English Second Language (ESL) students, fail or not understand mathematics. Several researchers have devoted time to investigate issues that relate to poor performance of students in mathematics (for example, Vale, Murray & Brown 2012). Vale *et al.* (2012) point out that, 'English second language students face an additional challenge in their National Curriculum Vocational (NCV) studies, namely that of learning and being assessed in a language other than home language'. This problem is exacerbated by the fact that the majority of South African students study mathematics in English; and most institutions of higher learning have English as their medium of instruction.

Apart from the problem of English Second Language (ESL) impacting negatively on the general performance of students, the language of mathematics is also viewed as problematic for many. In line with this view, Dempster and Reddy (2007) advance a convincing argument when suggesting that, learning of scientific subjects (Mathematics included) requires students' proficiency in both (i) language of mathematics and the (ii) language of instruction. In this respect, Jawahar and Dempster (2013:1429) submit that ESL students tend to be faced by an additional challenge, as they tend not to be in a position to master both the former and the latter. Arguably, English as the medium of instruction (in this case) mediates the comprehension and comprehensibility of the language of mathematics and, as such, one senses that the language of mathematics and the language of instruction may not be mutually exclusive, particularly in an environment where English is the medium of instruction. Put differently, mastery of English is, in some sense, a prerequisite for understanding and making sense of the language of mathematics. Mathematical discourse is a concern of this paper. This discourse consists of rules, symbols and formulae, some of which are derived from foreign languages. As a result of that, students tend to demonstrate poor interpretation of symbols in their calculations of mathematical problems (Siyepu, 2013b). Cangelosi *et al.* (2013:71) point out that students memorise algebraic rules with no conceptual understanding attached to these concepts in a mathematics discourse. They (2013) also note that many students have difficulty keeping track of and applying the rules appropriately; and they (2013) add that students often misinterpret -9^2 as equal $(-9)^2$ not $-(9)^2$. For example, some students interpret 2^{-3} as $2^{1/3}$ instead of $\frac{1}{2^3}$. The point being made is that an under-developed conception of additive and multiplicative inverse is at the root of these errors; and it is for that reason, among others, that the language used and the difficulty in interpreting notation and grouping may hinder students' progression *(ibid).*

The study of calculus, with its fundamental concepts, requires students to interpret mathematical signs, symbols and rules appropriately (Gray *et al.* 2009). Students' difficulties in the learning of calculus are well documented; but there seems to be scanty research (to the best of our knowledge) that focuses on making sense of symbols, rules and formulae in a calculus classroom. The inadequate interpretation, or misinterpretation, of symbols results in the failure of students to establish the interconnectedness of their existing mathematical knowledge with the new knowledge to be acquired; and this tends to have some negative implications for their success. This article explores students' interpretations of symbols, rules and formulae in a calculus classroom in their learning of differentiation in their first year in a university classroom. The primary purpose of this study was to explore and understand how students interpret symbols, rules and formulae as they attempt to attach meaning to and make sense of mathematical discourse. The intention was to gain this understanding through identifying students' errors in their written text and classroom interactions. Attention was also paid to addressing the identified errors in classroom discussion, with a view to improving students' understanding and, ultimately, their success.

This study sought to reveal errors displayed by students registered for mathematics in their learning of differentiation in their 1st year level in a university classroom. Explicitly, this study sought to answer questions such as:

1. What are errors displayed by students registered for mathematics in their learning of differentiation?

2. What strategies that are used by the lecturer to eliminate students' errors in their learning of differentiation?

3. How do these strategies improve learners' understanding of differentiation?

Theoretical Framework

This study is based on Sfard's (2007) Commognitive framework. Within this framework, *thinking* is defined as an individualisation of interpersonal communication, although not necessarily verbal. To emphasise the unity of cognitive processes and communication, the word *commognition*, a combination of the two (that is, cognitive processes and communication), is used to name the framework. The Commognitive framework is an analytical framework of the communicational approach to cognition, which could be perceived as including both cognitive and socio-cultural approaches. These approaches view learning as a process of becoming a participant in a certain distinct discourse. Discourse is considered a special type of communication, made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions. Sfard (2001:28) asserts that the concept of discourse refers to any specific act of communication, 'whether diachronic or synchronic, whether with others or with oneself, whether predominantly verbal or with the help of other symbolic systems'. Sfard (2007:573) adds: '…the different types of communication that bring some people together while excluding some others are called discourses'. She further explains that a discourse counts as mathematical if it features mathematical words, such as those related to quantities and shapes. This study deals with calculus, focusing on students' interpretations of calculus concepts and their notations, symbols and rules.

Sfard (2007:568) makes the following claim:

> If an interpretive framework is to pass the test, studies guided by this framework must be able to cope with the following issues:
>
> (i) *Focus on the object of learning:* In the case under study, what kind of change is supposed to occur as a result of learning?
>
> (ii) *Focus on the process:* How do the students and the lecturer work towards this change?
>
> (iii) *Focus on the outcome:* Has the expected change occurred?

Due to the concern of educationists and researchers about students' difficulties learning mathematical concepts, a shift from acquisitionist to participationist emerged. Sfard (2007:570) asserts that 'the participationist account comes to the rescue not only by offering a different answer to the question of how humans develop, but also by altering the conception of what it is that develops'. In a Commognitive approach, Sfard (2007:573) states that, 'in any academic discipline such as mathematics a form of discourse made distinct by four characteristics may be considered'. These comprise (a) words and their uses; (b) visual mediators; (c) endorsed narratives; and (d) routines, as detailed below.

Words and their uses: In any professional discourse, there are words and their uses that comprise the unique vocabulary of that particular discipline. Mathematics has its own language. It shares words with ordinary English but these have a different meaning in the context of mathematics. Mathematics 'register' is defined as the meanings belonging to the natural language used in mathematics (Cuevas 1984). Halliday (1975) asserts that a mathematics register has the following components:

(i) Natural language words reinterpreted in the context of mathematics, such as functions, root, derivative, product, chain, composite and differentiation.
(ii) Locutions, such as the square on the hypotenuse and least common multiple.
(iii) Terms formed from the combining elements of Greek and Latin words, such as parabola, denominator, coefficient and asymptotic.

In addition to vocabulary, a mathematics register also includes styles of meaning and ways of presenting arguments within the context of mathematics. In a calculus classroom, special words or concepts with their notations or symbols should be introduced with care to distinguish between their meaning in everyday English and in a mathematics context.

Visual mediators are the means by which participants of discourses identify the object of their talk and coordinate their communication. Mathematical discourses often involve symbolic artefacts, created specifically for the sake of a particular form of communication. The most common examples include mathematical notations, symbols, rules and formulae. In order to flesh out this notion, the following mathematical notations, symbols, rules and formulae are worthy of note.

(a) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$

(b) Power Rule *is described by the equation below:* $\frac{d}{dx}(x^n) = nx^{n-1}$ *for* n *is equal to any constant.*

(c) Constant Rule: $\frac{d}{dx}(c) = 0$. *The derivative of a constant is zero.*

(d) Constant Multiple Rule: The derivative of a constant multiplied by a function is the constant multiplied by the derivative of the original function: $\frac{d}{dx}(a \cdot f(x)) = a \cdot \frac{d}{dx}(f(x))$

(e) Sum/Difference Rules: *The derivative of the sum of two functions is the sum of the derivatives of the two functions:* $\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$

(f) Product Rule: *The derivative of the product of two functions is described by the equation here* $\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot \frac{d}{dx}(g(x)) + g(x) \cdot \frac{d}{dx}(f(x))$

(g) Quotient Rule: *The derivative of the quotient of two functions is described by the equation here* $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{\frac{d}{dx}(f(x)) \cdot g(x) - f(x) \frac{d}{dx} g(x)}{(g(x))^2}$

(h) Chain Rule: The chain rule is used to differentiate composite functions. As such, it is a vital tool for differentiating most functions of a certain complexity.

It states: $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$.

(i) Differentiation of exponential functions: The derivative of $y = b^x$ is given by $\frac{d}{dx} = b^x \ln x$

(j) Logarithmic differentiation provides a way to differentiate functions such as $y = x^x$; $y = x^{(ex)}$; $y = (\sin x)^{x^3}$ and $y = x^{(x^{(x^4)})}$. These functions require expert algebra skills and careful use of the following unpopular, but well-known, properties of logarithms. Though the following properties and methods are true for a logarithm of any base, only the natural logarithm (base e, where $e \approx 2{,}718281828.....$) are considered. Notably, this study uses the natural logarithm $(\ln)$ and thus students' understanding of logarithmic differentiation requires them to master the following properties of natural logarithms.

(a) $\ln e = 1$
(b) $\ln e^x = x$
(c) $\ln y^x = x \ln y$
(d) $\ln(xy) = \ln x + \ln y$
(e) $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$

All of the above mathematical formulae, rules, symbols and notations are the basic requirements for any student in order to develop interconnectedness of differentiation.

Endorsed narratives are facts and ideas that are true in conventional mathematical knowledge. These entail definitions, axioms, theorems and formulae. In the case of mathematical discourse, the consensually endorsed narratives are known as mathematical theories, which include the rules and formulae discussed above in relation to visual mediation (Sfard 2007). Making sense of rules, notations and formulae that are applicable in a calculus classroom is considered part of this study.

Routines are procedures that are applied as repeated steps to reach a solution in a mathematical problem. Routines are helpful in learning a new discourse, as our ability to act in new situations often depends on recalling one's or others' past experiences (Tabach & Nachlieli 2011). An example of a routine in calculus is the application of the first principle of differentiation to find the derivative $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

Literature Review

Students' understanding of mathematics remains a problem, despite several attempts by teachers, lecturers and researchers to develop viable strategies that could be applied in the classroom to enable students to improve their performance. The language of mathematics is regarded as the main cause of this inadequate understanding. Students' tendency to make errors when learning mathematics seems to be related to their poor interpretation of concepts. Students' displayed errors are based on these factors: over-generalisation; ignorance of rule restrictions; incomplete application of rules; and misinterpretation of concepts (Cuevas 1984).

Mathematical Discourse

Adler (2001) outlines language scenarios that are relevant to the South African situation, with specific reference to the language culture in a Mathematics classroom. Firstly, in the urban-suburban areas where there is a strong English environment, many different main languages are found. Secondly, in urban or township contexts, there is less English in the environment, with the presence of a strong regional language and different indigenous languages. Thirdly, there are foreign language situations where the learners mainly hear English at school but most of the learners use the same main language, i.e. not English. The two latter scenarios seem to have a direct bearing on the current study, because a reasonable number of students who participated in the study hail from the townships of the Western Cape, whereas all others originate from the deep rural areas of the Eastern Cape. As pointed out earlier in the discussion, in South Africa, English is used as the language of instruction from the middle level of schooling to tertiary level. Subjects such as Mathematics are taught to students, the majority of whom have limited English proficiency. These students struggle to understand Mathematics lessons owing to, amongst other factors, poor everyday English vocabulary and a poor understanding of mathematical discourse.

English language speakers select their vocabulary to convey meaning in a particular context. This also applies within the discourse of mathematics, where mathematical concepts have special meanings. Cuevas (1984:137) asserts that 'mathematical concepts are viewed as the result of the students' experience, with language facilitating the students' conceptual development through discussion and instruction'. He further elaborates that language is applied to the content of mathematics in the representation of experience through mathematical notation. The use of signs, symbols, rules and formulae in mathematics confuses students, as they struggle to access the meaning of this terminology.

The main task for Mathematics lecturers is to help students to make sense of all these mathematical statements. This may be done by means of semiotic activity. Semiotic activity is defined as the activity of investigating the relationship between sign and meaning, as well as improving the existing relationship between sign and meaning (Van Oers 1997). This suggests that lecturers should focus on making meaning of mathematical vocabulary and procedures in their teaching of mathematics.

Communication in a Mathematics Classroom

The poor performance of students in Mathematics is perhaps owing to lecturers' teaching approaches in Mathematics classrooms. Studies show that instruction remains lecturer-centred, with greater emphasis placed on lecturing than on helping students to think critically and apply their knowledge to real-world situations (Cobb *et al.* 1992). Several researchers propose the development of an inquiry-based form of mathematics instruction. In an inquiry-based environment, learning is viewed as an active, constructive activity in which students are encouraged to explore, develop conjectures and solve problems (Wachira *et al.* 2013). Students are encouraged to discuss and communicate their ideas and results, often within small, cooperative groups, as well as with their lecturers. The National Council of Teachers of Mathematics (2000) suggests that instruction should provide students with opportunities to engage in mathematical inquiry and meaning-making through discourse; and lecturers should encourage this process by remaining flexible and responsive to students' response and feedback.

Wachira, Pourdavood & Skitzki (2013:2) claim that 'a crucial aspect of a classroom in which students are actively engaged, is to focus on classroom discourse'. They define discourse as purposeful talk on a mathematics subject in which there are contributions and interactions that unpack mathematical concepts among students. They elaborate that discourse does not only promote the development of shared understandings and new insights, but also contributes to deeper analyses of mathematics by the lecturers. They claim that a key element of discourse is the need to use mathematics language and articulate mathematics concepts in order to learn both the language and the concepts. This study employs high discourse classrooms. Imm and Stylianou (2012:131) relate to this notion when pointing out that in high discourse classrooms lecturers prioritise exchange of ideas among the students, and the exchange of ideas should be a purposeful mathematical conversation.

Communication is important in developing mathematical understanding (Steele 2001). Steele (2001) explains that, within a socio-cultural perspective, students exchange ideas with one another and listen actively to one another's views. This creates mutual understanding based on culturally established mathematical practices. Ryve, Nilsson and Pettersson (2013) suggest that students need to enter each other's universe of thought in classroom interaction. Communication should be effective: it is effective if it assists students to gain insight into what is being discussed about a particular topic being studied. In a high discourse Mathematics classroom, students are assigned tasks to calculate and communicate their thinking with others. In this way, students express their understanding and interpretation of their mathematical tasks. The next section discusses students' difficulties in learning calculus.

Students' Conceptual Difficulties in Understanding Calculus

Making sense of mathematics gives students pleasure, confidence and a willingness to tackle new problems (Tall, 2013). He (2013) also notes that the long-term growth of mathematical thinking is improved for those who have a 'sense of relationships' that guides their thinking. Additionally, for students to develop an understanding of mathematics, there should be a sensible approach that takes account of the structures and increasing levels of sophistication involved as learning progresses from sense through perception, then through the relationships of operations and a developing sense of reasoning (Tall, 2013). In other words, students should be able to connect meanings of symbols, rules and formulae in their learning of differential calculus.

Some students' understanding of calculus is hindered by their lack of ability to make sense of calculus concepts (White & Mitchelmore 1996). Sfard (2008:111) defines a concept as 'a symbol together with its uses'. White and Mitchelmore (1996) argue that the main inhibiting factor to success in calculus seems to be an underdeveloped concept of a variable. Consequently, students suffer from a *manipulation focus* where they base decisions about which procedures to apply to the given symbols and ignore the meaning behind the symbols. They (1996) argue that being able to symbolise derivatives involves forming relationships between concepts and should therefore be indicative of conceptual knowledge.

Tall (1993:2) claims that, 'whichever way the calculus is approached, there seem to be difficult concepts which seem to cause problems no matter how they are taught'. , He continues to argue that, 'when students meet difficulties, a dominant strategy for coping is to concentrate on the procedural aspects that are usually set in set examinations' (Tall 1993:4). He provides a list of examples of difficulties with calculus that are normally displayed by students (Tall 1993:6):

- Leibniz notation $\frac{dy}{dx}$ proves to be almost indispensable in calculus, yet it causes serious conceptual problems. Students fail to understand whether $\frac{dy}{dx}$ is a fraction or a single symbol. In the same vein, students fail to understand the relationship between the dx in $\frac{dy}{dx}$ and the dx in $\int f(x)dx$. Another query he raises is, 'can the du be cancelled in the equation $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$?'

- Difficulties in selecting and using appropriate representations are known to be widespread.
- Algebraic manipulation is the preferred mode of operation for many students. Students who take calculus with little knowledge of manipulating polynomials and trigonometric formulae tend to experience difficulties in simplification processes in solutions of differentiation.

Research Design and Methodology

This is a qualitative case study located within the interpretive paradigm. Shank (2002:5) defines qualitative research as 'a form of organised empirical inquiry into meaning'. Denzin and Lincoln (2000:3) further maintain that qualitative research involves an *interpretive and naturalistic* approach and thus '…qualitative researchers study things in their natural settings, attempting to make sense of or to interpret, phenomena in terms of the meanings people bring to them'. This study focuses on making sense of mathematical discourse with the intention of developing meaning and understanding the concept of differentiation among the first year students. The purpose is to attempt to find viable solutions to student difficulties in learning of differentiation in a calculus classroom and improve their success. This study reports on implications of using mathematical classroom discourse during social interactions to unpack calculus concepts.

Case Study

This study adopts a case study design approach. A case study is an empirical inquiry that investigates a phenomenon within its real life context (Yin 2009).

Merriam (1988) defines a qualitative case study as an intensive, holistic description and analysis of a single entity. Case studies are particularistic, which means that a case study focuses on a particular situation. They rely heavily on inductive reasoning in handling multiple data sources.

Research Participants

The research participants were thirty students who enrolled for Chemical Engineering for first semester level in an Extended Curriculum Programme (ECP) during the 2009 academic year. They were borderline cases, meaning that they did not necessarily meet the minimum requirements for entry into the main engineering stream.

Data Collection

The data set for this study was collected from students' written work, and from audio and video recordings. The latter offer the advantage of dense, authentic data. In a case study approach, researchers seek to study participants in real situations, doing real activities. To substantiate this view, DuFon (2002:43) argues that: Audio and video recordings can provide researchers and other interested parties with more contextual data. Further they give a complete sense of who the participants are, and acquaint people with the setting in which the participants function and the type of activities they engage in, and the nature of these activities. Audio and video recordings are permanent; they allow researchers and other interested parties to experience an event repeatedly by listening to and viewing these recordings as many times as necessary. Replaying an event allows researchers more time to reflect on the data before drawing conclusions. The audio and video recordings supported the data collection process through bringing a high level of detail regarding the interactions between the researcher and students (Pelling & Renard 1999).

In this study, students and the researchers gathered in a lecture room so that the students could demonstrate their interpretations of calculus activities. They shared their understanding and interpretations to reach consensus about the correct interpretation of rules and symbols to make sense of their learning of differentiation. Figure 1 below shows activities which the participants discussed.

Question 1

Differentiate the following functions and leave your answer in simplest form.

1.1 $y = \dfrac{\sec^3 x^2}{\tan x}$

1.2 $y = 3^{2x+5} \ln(7x-9)$

Question 2

Differentiate the following functions implicitly and leave your answer in simplest form.

2.1 $x + y = \ln(x^2 + y^2)$

2.2 $e^{y+x} - e^x = e + e^y$

Question 3

Find the derivatives of the following functions and simplify where possible

3.1 $y = x^3 e^{2x+3} \sqrt{\cos x}$

3.2 $y = \sec^2 x e^{-\tan^2 x}$

Figure 1: The test given to the students

Data Analysis

Data analysis was conducted in multiple stages. Upon completion of marking students' written tests, the lecturer carried out item-by-item analysis by examining students' responses for each item. The students' scripts were sorted and the scripts that displayed similar errors were grouped.

Audio and video recordings from the classroom lessons were transcribed. The focus was on errors displayed by the students in their calculations on a whiteboard.

The data analysis also focused on the classroom interactions and exchange of ideas among the students and the lecturer. Data analysis revealed that some students misinterpret rules and symbols in their use of differentiation rules. During mathematical discourse in the classroom, this misinterpretation drew the attention of both the lecturer and the students.

Classroom interactions and exchange of ideas involves students and lecturers' evoking one another's understanding of differentiation concepts as students explain their thinking in the process of calculation.

Discussion of Results

This study considered student errors in line with the conceptual framework discussed below:

- Conceptual errors, according to Kiat (2005), are evident in a failure to grasp the concepts in a problem and a failure to appreciate the relationships in a problem. From a Commognitive stance, conceptual errors are connected with poor understanding of words and their uses.

- Interpretation errors, according to Olivier (1989), occur when students wrongly interpret a concept due to over-generalisation of the existing schema. From a Commognitive point, interpretation errors are interconnected with poor understanding of words and their uses. To be exact, students might know mathematical formulae but unable to apply them appropriately.

- Linear extrapolation errors occur when students over-generalise the property $f(a + b) = f(a) + f(b)$, which applies only when f is a linear function, to the form $f(a * b) = f(a) * f(b)$, where f is any function and * any operation (Matz, 1980). From a Commognitive standpoint, linear extrapolation errors are related narratives as students fail to understand the restrictions of the rules.

- Procedural errors, according to Kiat (2005), occur when students fail to carry out manipulations or algorithms, although concepts are understood. From a Commognitive position, procedural errors are related to routines where students fail to follow repetitive patterns in interlocutors' actions.

- Arbitrary errors, according to Orton (1983), occur when students behave illogically and fail to take account of the constraints laid down in what was given. A commognitive justification is where students have poor visual mediators. In the case of differentiation where students do not know the appropriate formulae to be applied.

The discussion of results is an attempt to reveal errors displayed by students in their written work. An explanation on how the errors displayed was addressed in classroom discussion. Of particular interest and importance was the students' improvement that seemed to take place as a result of this teaching and learning initiative.

Conceptual Errors Displayed by Students

Conceptual errors occur owing to a failure to grasp the concepts involved in the problem, or failure to appreciate the relationships involved in the problem.

Table 1 shows a conceptual error and its description in an exponential logarithmic function

Type of error	Description
Conceptual errors	Conceptual errors occur owing to a failure to grasp the concepts involved in the problem or failure to appreciate the relationships involved in the problem. For example, it did not occur to some students that $\ln e = 1$

In activities attempted by eight students, the students did not register that $\ln e = 1$, hence they applied the product rule to differentiate $\tan^2 x \ln e$. They also could not see the relationship of the concept of a natural logarithm ($\ln x$) and the cosine function given in the problem 3.1. Hence they differentiated $\frac{1}{2}\ln\cos x$ as $\frac{1}{2}\ln\cos x \cdot -\sin x + \frac{\frac{1}{2} - \sin x}{\cos x}$. One student could not see that $\sqrt{\cos x} = (\cos x)^{\frac{1}{2}} \neq \cos x^{\frac{1}{2}}$, and she differentiated $\ln\sqrt{\cos x}$ incorrectly. For example, her solution was as follows:

$$\frac{d}{dx}(\ln\sqrt{\cos x})$$

$= \frac{1}{\sqrt{\cos x}} \cdot (-\sin x)$.

Two students showed errors of simplification of trigonometric functions, as they did not apply the Lowest Common Denominator (LCD) correctly.
One student showed errors in differentiation of trigonometric functions when incorporated with logarithmic functions. His solution indicated that he could not distinguish between the power rule and the logarithmic differentiation. He differentiated $\ln x^3$ as $3\ln x^2$ instead of $\frac{3}{x}$. He wrote that the derivative of $y = \ln e^{2x+3}$ is $\frac{dy}{dx} = 2x + 3\ln e \cdot 1$.

Students also differentiated $y = \ln\sqrt{\cos x}$ as $\frac{1}{2\sqrt{\cos x}} \cdot \frac{-\sin x}{1}$ instead of $\frac{1}{\sqrt{\cos x}} \frac{1}{2\sqrt{\cos x}} \cdot -\sin x$. One student did not substitute $\frac{-\sin x}{2\cos x}$ with $\frac{-\tan x}{2}$. This indicates poor understanding of trigonometric identities.

One student differentiated the following function as: $y = \ln\sec^2 x$ $= 2\ln\sec x \cdot \frac{1}{\sec x} \cdot \sec x \tan x = 2\ln\sec x \tan x$ instead of writing $\ln\sec^2 x$ as $2\ln\sec x$ first and then differentiating $2\ln\sec x$ as $2 \cdot \frac{1}{\sec x} \cdot \sec x \tan x = 2\tan x$.

Another student showed a poor understanding of the chain rule as she differentiated $\ln\sec^2 x$ as $\ln\sec^2 x \cdot 2\sec x \cdot \sec x \tan x$. She also differentiated $-\tan 2x$ as $\sec^2 x(-\tan 2x)$, instead of $-\sec^2 x \cdot 2 = -2\sec^2 x$.

Another student differentiated $\ln\sec^2 x + \ln e^{-\tan^2 x}$ incorrectly, as she wrote that $\frac{y'}{y} = \frac{1}{\sec^2 x} \cdot 2\sec x \cdot \cos ecx \tan x + (-\tan^2 x)\ln e$ is the derivative of $\ln \sec^2 x + \ln e^{-\tan^2 x}$.

This solution indicated that this student did not know that the derivative of $\sec x$ is $\sec x \tan x$.

A student wrote $\frac{1}{\sin x}$ as the derivative of $\sec x$; another student wrote $\sec x$ as the derivative of $\tan x$, while yet another student differentiated $-\tan^2 x$ as $-2\tan x \cdot -\sec^2 x$. The error is to write a minus sign in front of $\sec^2 x$.

A student showed a poor understanding of the chain rule as they differentiated $-\tan^2 x$ as $-\tan^2 x \sec^2 x$ instead of $-2\tan x \sec^2 x$. Yet another student wrote that the derivative of $\sec^2 x$ is $-\cos ec^2 x$.

Another student wrote that the derivative of $-\tan^2 x$ is $-\tan^2 x \cdot 2\tan x \sec^2 x$. One student wrote that the derivative of $\tan x$ is $\cot x$. Two students wrote that the derivative of $-\tan^2 x$ is $\ln \sec x \cdot 0 - \sec^2 x$. Two students did not know how to differentiate a composite function from a trigonometric function such as $-\tan^2 x$. One of these two students wrote $-\sec x$ as the derivative of $-\tan^2 x$. The other wrote $-(\sec^2 x)^2 \sec^2 x$ as the derivative of $-\tan^2 x$, and also wrote that the derivative of $\sec^2 x$ is $(\sec x \tan x)^2$. This error originated from the algebraic over-generalisation that if $a = b$ then $a^2 = b^2$.

Interpretation Errors Displayed by Students

Interpretation errors arise when students fail to interpret the nature of the problem correctly owing to over-generalisation of certain mathematical rules involved in the problem.

Table 2 shows an interpretation error and its description in an exponential logarithmic function

Type of error	Description
Interpretation error	Students fail to interpret the nature of the problem correctly due to over-generalisation of certain mathematical rules. For example, some students wrote that the derivative of $y=\sec^2 x$ is $\frac{dy}{dx}=\tan x$.

Two students could not remember the appropriate procedure to differentiate $y=3^{2x+5}$ in $y=3^{2x+5}\ln(7x-9)$. One student applied the power rule instead of the exponential rule. He showed confusion in differentiation of the following functions, $y=a^x; y=x^n$ and the chain rule. Another student applied logarithmic differentiation to differentiate $y=3^{2x+5}\ln(7x-9)$. Figure 2 shows an example of the kind of interpretation error that the students displayed in the differentiation of the function $y=3^{2x+5}\ln(7x-9)$.

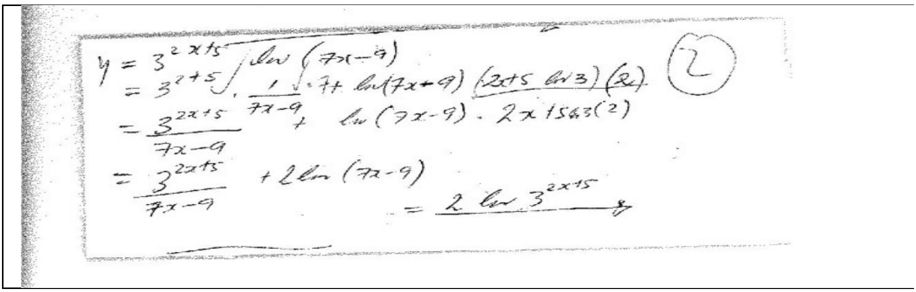

Figure 2: An interpretation error displayed in differentiation

Fourteen students fused two functions into one function in the differentiation of $y=x^3e^{2x+3}\sqrt{\cos x}$. They treated x^3e^{2x+3} as the first function and $\sqrt{\cos x}$ as the second function. One student could not remember the derivative of $y = \sec^2 x$ and, as a result, she wrote that the derivative of $y = \sec^2 x$ is $\frac{dy}{dx} = \tan x$. This error originated from over-generalisation of the symmetric property, which states that for any quantities a and b, if $a = b$, then $b = a$. This is not so in the case of derivatives.

One student wrote that the derivative of $y = -(\sec^2 x)^2 \sec^2 x$ is $\frac{dy}{dx} = -\tan^2 x$ and also wrote that the derivative of $y = \sec^2 x$ is $\frac{dy}{dx} = (\sec x \tan x)^2$. This originated from the algebraic over-generalisation, such as if $a = b$ then $a^2 = b^2$.

Procedural Errors Displayed by Students

Procedural errors occur when students fail to carry out manipulations or algorithms, although they understand concepts in the problem.

Table 3 shows a procedural error and its description in an algebraic function

Type of error	Description
Procedural error	Procedural errors arise when students fail to carry out manipulations or algorithms, although they understand concepts in the problem. For example, one student could not multiply $-\sec^2 x(\sec^3 x^2)$ correctly.

Ten students demonstrated difficulty in simplification of trigonometric functions. One student could not apply the appropriate procedure in differentiation of $y = \frac{\sec^3 x^2}{\tan x}$. He did not consider that he needed to apply the quotient rule. Figure 3 below shows an example of a procedural error that the students displayed in the differentiation of a trigonometric function.

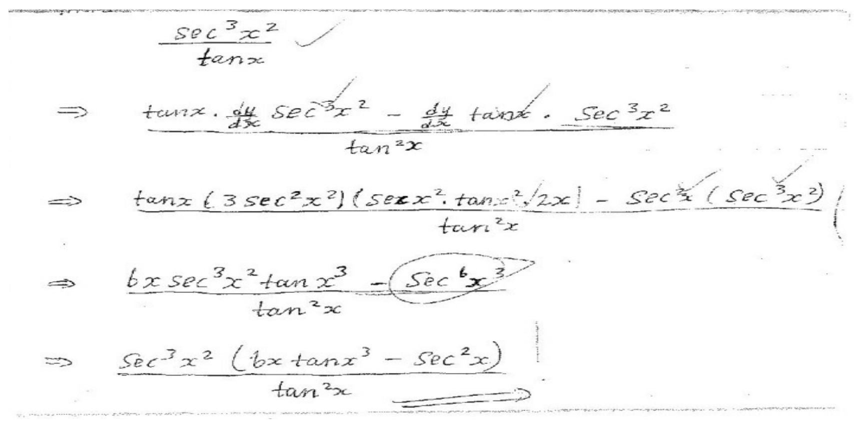

Figure 3: A procedural error displayed in differentiation of a trigonometric function

One student showed poor understanding of identities as he wrote $\cot\theta = \tan\theta$. As a result, he substituted $\cot\theta$ with $\tan\theta$. One student failed to multiply radical trigonometric functions correctly. He manipulated

$\frac{1}{\sqrt{\cos x}} \cdot \frac{1}{2\sqrt{\cos x}} \cdot \frac{-\sin x}{1}$ incorrectly. As a result, he obtained $\frac{-\sin x}{\sqrt{\cos x}}$

instead of $\frac{-\sin x}{2\cos x}$. One student failed to apply the LCD correctly in

$y = x^3 e^{2x+3}\sqrt{\cos x}$. Two students differentiated $y = 2x + 3\ln e$ incorrectly.

They did remember to apply the sum rule.

They also treated $y = 2x + 3\ln e$ as if it required the application of the product rule, treating $2x + 3$ as the first function and $\ln e$ as the second function. They wrote that the derivative of $(2x+3)\ln e$ is $2\ln e$.

Linear Extrapolation Errors Displayed by the Students in the Three Tests

Linear extrapolation errors happen through an over-generalisation of the property $f(a+b) = f(a) + f(b)$, which applies only when f is a linear function. Linear extrapolation errors may be regarded as a subset of an interpretation error, as they occur due to poor interpretation of certain mathematical rules (Siyepu 2013b).

Table 4 shows a linear extrapolation error and its description in an implicit function

Type of error	Description
Linear extrapolation errors	Linear extrapolation errors happen through an over-generalisation of the property $f(a+b) = f(a) + f(b)$, which applies only when f is a linear function. For example, one student calculated $x + y = \ln(x^2 + y^2)$ as $\ln x + \ln y = \ln x^2 + \ln y^2$

Five students demonstrated linear extrapolation error as they multiplied an algebraic expression by the symbol of a natural logarithm $\ln$ and differentiated the expression by using the sum and difference rule. Their error shows an over-generalisation of the distributed property as they treated the logarithmic function $\ln x$ as an ordinary variable. Figure 4 below shows an example of a linear extrapolation error that the students displayed in the differentiation of an implicit function.

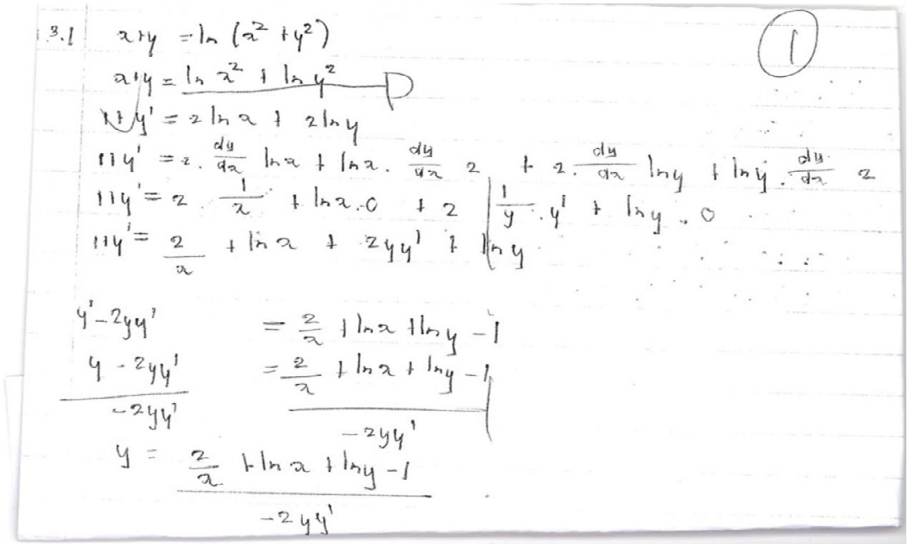

Figure 4: A linear extrapolation error displayed in implicit differentiation

Arbitrary Errors Displayed by Students

Arbitrary errors arise when students behave illogically and fail to take account of the constraints laid down in what is given.

Table 5 shows an arbitrary error and its description in a trigonometric function

Type of error	Description
Arbitrary error	Arbitrary errors arise when students behave arbitrarily and fail to take account of the constraints laid down in what is given. For example, one student rewrote $y = \cos^4(5x^2)$ as $y = \cos(5x^2)^4$.

Two students did not show any logic in their differentiation of $y = x^3.e^{2x+3}\sqrt{\cos x}$. They did not apply logarithmic differentiation correctly to obtain $\ln y = \ln x^3 + \ln e^{2x+3} + \ln\sqrt{\cos x}$. Two students only transcribed the problem without any attempt to do calculations.
Two other students transcribed the problem incorrectly and also showed illogical steps in their calculations of $y = \sec^2 xe^{-\tan^2 x}$. The first one transcribed the problem as $y = \sec^2 xe^{-\tan x}$ instead of $y = \sec^2 xe^{-\tan^2 x}$. The second one transcribed the problem as $y = \sec^2 xe^{-\tan 2x}$ instead of $y = \sec^2 xe^{-\tan^2 x}$. One student left $(2x+3)\ln e$ without differentiating it.

Students' Responses as Reflected in the Audio and Video Recordings

In the audio and video recordings, three students showed their solutions on a whiteboard, whilst representing their groups. The students demonstrated errors that had already been identified in their written work. The lecturer intervened by explaining appropriate procedures, describing concepts that had been interpreted incorrectly as the students were explaining their understanding of the derivatives of various functions.

One student argued that her understanding was that logarithmic differentiation is applied only when the base of the function is in the form of a variable, and that the index is also a variable. In the case of $y = x^3 e^{2x+3}\sqrt{\cos x}$, all the terms are not in a transcendental form. This student's question indicated that she had confused the application of the chain rule with the application of the logarithmic differentiation rule. This problem contains three functions whilst the students were familiar with differentiation of two functions (the latter makes it easy for them to apply the product rule).

One student questioned why we do not apply the power rule to differentiate $y = \ln x^3$. His question might have been asked for purposes of clarity, or may show that he did not know the difference between the power rule and logarithmic differentiation. This question may thus symbolise poor conceptualisation.

In response, the lecturer explained that, in the case of a natural logarithmic function, one does not apply the power rule. In the explanation, the lecturer used examples of x^3 and $\ln x^3$, showing techniques of differentiating these two different functions. He further explained that the first function x^3 requires the application of the power rule with its derivative equal to $3x^2$; and that the second function $\ln x^3$ requires the application of logarithmic differentiation to obtain its derivative, which is equal to $\frac{f'(x)}{f(x)} = \frac{3x^2}{x^3} = \frac{3}{x}$. This intervention assisted

the student working on audio and video recordings to rectify her mistake. The lecturer also intervened by correcting errors as he explained the appropriate procedure involved in cancelling trigonometric functions that fall under addition.

One student raised a question, which reflected a conceptual error. He wanted to know whether it is appropriate to substitute $\sec^2 x$ with $\tan x$. This question showed that the student did not understand that, although the derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$, the derivative of $\sec^2 x$ is not $\tan x$

The lecturer explained the appropriate procedure of obtaining the derivative of $y = \sec^2 x$. One student suggested a further simplification of

$y' = [2\tan x - 2\tan^2 x \sec x]\sec^2 xe^{-\tan^2 x}$ to

$y' = 2\tan x[1 - \sec^2 x]\sec^2 xe^{-\tan^2 x}$.

The lecturer explained that to remove a common factor would be an undesirable closure, as it is the opposite of simplification.

A student requested the use of the product rule to differentiate this problem. The same student attempted to address the problem by using the product rule in audio and video recorded observations. The student showed that she had poor understanding of the standard derivatives.

Implications for Teaching and Learning

This section deals with implications of this study in the light of a Commognitive framework. The goal of any teaching is to develop understanding among the students in order to gain knowledge and insight on the topic being studied.

One student applied the power rule instead of exponential rule. In the case understudy the kind of change to occur as result of learning the students should be able to differentiate the differentiation problem that requires application of power rule and/or exponential function. He showed confusion in differentiation of the following functions $y = a^x$; $y = x^n$ and the chain rule. A commognitive justification is that in order for the teacher and students to focus on the process to occur to work towards a change, emphasis should be done in classroom activities and discussions to explain clearly the difference between power rule and exponential rule.

In the context of this study, we could confidently argue that 'understanding' has taken place, as Hiebert and Carpenter (1992:67) substantiate:

> A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections.

Firstly, the students' understanding of differentiation was determined by their ability to see the relationship of the concepts involved in the problem and apply appropriate procedures to obtain the solution. This was achieved not only through engaging students with learning activities of a mathematical nature, but also through the aid of the more competent students, along with the lecturer guidance.

Secondly, in terms of students who confused the product rule with the logarithmic rule, their solutions showed that they were not aware of $\ln x$ as a natural logarithm; they assumed that $\ln x$ was any other variable. As a result, they could not apply properties of a natural logarithm; instead they applied the product rule. For these students, there seemed to be no difference between $\ln(\sin x + 2)$ and $x(\sin x + 2)$. This was also evident as the lecturer assigned the students with activities that elicited application of a natural logarithm ($\ln x$). It stands to reason therefore that the concept of a natural logarithm and its properties should be further demystified for students to comprehend. In addition, the difference between natural logarithmic functions and algebraic functions should be made explicit in the process of imparting knowledge.

The kind of change in endorsement routines to occur as a result of learning is that students should be able to know the nature of differentiation problem that requires application of a product rule. Specifically, knowing that a product rule is applied when two or more functions are joined by a multiplication sign(s). A commognitive justification is that students should be able to know that logarithmic differentiation is appropriate in the case of functions such as $y = x^x$ where the base is a variable as well as the exponent is also a variable.

Thirdly, it has become clear in this study that the process of engaging students with learning activities, along with tapping their independent thinking to some degree, reinforced their understanding. Their understanding became evident when they were assigned to solve mathematical problems independently. This was further demonstrated not only by being able to identify interrelationships between the concepts and appropriate procedures, but also through displaying confidence and working independently throughout the activity, without requiring any form of assistance from the lecturer or from their more capable peers.

Fourthly and finally, for the students who were not familiar with the differentiation of the function $y = a^x$ where a is a constant and x is a variable, the lecturer designed learning activities that would capitalise these errors so that students would be in a position to realise their errors and misconceptions without the lecturer's intervention. As a result, they became independent in terms of their thinking, such that it did occur to them that the two functions $y = x^n$ and $y = a^x$ are different. It also transpired that these students could remember the restriction of the rule $\frac{dy}{dx} = nx^{n-1}$, that is n is strictly a constant. At the same time, they were able to remember that the derivative of the function $y = a^x$ is $\frac{dy}{dx} = a^x \ln a$.

A commognitive justification is that students should understand the words and their uses to solve any mathematical problem correctly. Once students understand the use of appropriate rules and procedures then they will be able to master differentiation.

Conclusion

The results of this study suggest that lecturers should identify students' errors in order to be able to design learning activities that may enhance students' understanding of derivatives of various functions. Errors displayed by students in this study mostly originated from their prior learning of mathematics and over-generalisation of certain mathematical rules. The students' prior learning had been dominated by rote learning of routines or procedures without their having made sense or meaning of these. As a result, they tended to apply rules hastily.

The use of the Commognitive framework is a utility and, as such, it emphasises individual attention to obtain students' explanations, discussions and elicit debates. This is an important spinoff as it also provides a sense of how and why students perform to reach their full potential, and what form of assistance they require to be in a position to devise viable solutions to their assigned mathematical problems. This, without doubt, requires investment in time and patience, if we are seriously concerned about enhancing the understanding and comprehension of students in as far mathematics is concerned, particularly when dealing with students enrolled in the Extended Curriculum Programme (ECP) and for whom English is an additional language.

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